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Best Approximation by Normal Operators

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The problem of approximating an arbitrary operator on Hilbert space by normal operators is studied, with special emphasis on those operators which admit zero as a best normal approximant.

INTRODUCTION

We are going to consider a certain kind of norm-extremal problem in the space $\mathscr{B}(H)$ of all bounded linear operators on a fixed Hilbert space H. Let us respectively denote the subsets of hermitian, positive, compact, and normal operators in $\mathscr{B}(H)$ by $\mathscr{H}(H)$, $\mathscr{P}(H)$, $\mathscr{C}(H)$, and $\mathscr{N}(H)$. Then it is known that each of the first three of these subsets is *proximinal* in $\mathscr{B}(H)$; that is, every operator in $\mathscr{B}(H)$ has a best approximation (or nearest point) from within $\mathscr{H}(H)$, $\mathscr{P}(H)$, and $\mathscr{C}(H)$. These results are established in [1], [5], and [2, 7], respectively. It is therefore natural to consider the analogous question for $\mathscr{N}(H)$: Does every operator admit a best normal approximation?

This question appears to be deeper than the others, perhaps in part because $\mathscr{N}(H)$ lacks any readily apparent geometric structure. We do know that it is closed, nowhere dense cone in $\mathscr{B}(H)$ [4], but unfortunately it is not convex. Consequently, most of the usual approximation-theoretic criteria do not apply. In addition, we can make no general assertions about metric properties of the norm in $\mathscr{N}(H)$, since $\mathscr{N}(H)$ is a kind of macrocosm for Banach spaces. More precisely, any separable Banach space can be isometrically embedded in $\mathscr{N}(H)$ (assuming of course that H is infinite dimensional).

In the present paper we make a modest beginning on the study of best normal approximation. After establishing upper and lower bounds for the distance between an arbitrary operator and $\mathscr{N}(H)$, we focus our attention on those (nonzero) operators T for which this distance is maximal. Such operators satisfy by definition the equation

$$\| T \| = \operatorname{dist}(T, \mathscr{N}(H)):$$

they shall be called here *antinormal* operators. The existence of such operators is strictly an infinite dimensional phenomenon: no compact operator can be antinormal. This is shown to follow from the result below that no invertible operator can be antinormal. We establish a sufficient condition for an operator to be antinormal, and note that all known examples of such operators (namely, the nonnormal maximal partial isometries) satisfy this condition. It is known ([6]; this example is generalized below) that not every partial isometry is antinormal but we conjecture that every subnormal partial isometry satisfies our sufficiency condition and is thereby antinormal.

1. Distance Estimates

In this section we give upper and lower bounds for the distance between an operator T and $\mathcal{N}(H)$ and show that, in general, these estimates are exact.

Theorem 1. *Let $T \in \mathcal{B}(H)$. Then*

$$\sup\{| \, \| T(x)\| - \| T^*(x)\| \, |; \| x \| = 1\} \leqslant 2 \operatorname{dist}(T, \mathcal{N}(H)) \leqslant \| T - T^* \|. \quad (1)$$

Proof. Since $(T + T^*)/2$ is a best approximation to T from $\mathcal{H}(H)$ [1], the right-hand side of (1) is just 2 dist$(T, \mathcal{H}(H))$, so the right-hand inequality follows from the inclusion $\mathcal{H}(H) \subset \mathcal{N}(H)$. The other inequality is proved (as was done in [6] for the case of partial isometries) by choosing a unit vector x and a normal operator N, an noting

$$\begin{aligned} | \, \| T(x)\| - \| T^*(x)\| \, | &\leqslant | \, \| T(x)\| - \| N(x)\| \, | + | \, \| N^*(x)\| - \| T^*(x)\| \, | \\ &\leqslant \|(T - N)(x)\| + \|(N^* - T^*)(x)\| \\ &\leqslant \| T - N \| + \| N^* - T^* \| = 2 \| T - N \|, \end{aligned}$$

Q.E.D.

To see that equality is possible in (1), we consider the operator T on a two-dimensional Hilbert space given by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. T is a partial isometry, and $T - T^* = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is unitary, so that $\| T - T^* \| = 1$. If also $x = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ then $\| T(x)\| - \| T^*(x)\| = 1$, and thus we have equality on both sides of (1). It follows that dist$(T, \mathcal{N}(H)) = \frac{1}{2}$ and it may be verified that $\frac{1}{2}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is a best normal approximant to T.

Let us further remark that equality on the right in (1) will be attained whenever T is antinormal, since any such operator satisfies dist$(T, \mathcal{N}(H)) =$ dist$(T, \mathcal{H}(H)) = \| T - T^*\|/2$. On the other hand, let T be an operator for which equality holds on the left in (1), and assume for simplicity that dist$(T, \mathcal{N}(H)) = 1$. Then if $S \in \mathcal{B}(K)$ satisfies dist$(S, \mathcal{N}(K)) \leqslant 1$ for some Hilbert space K, we have dist$(S \oplus T, (K \oplus H)) = 1$, so that $S \oplus T$ is also an operator for which equality is attained on the left in (1).

2. Antinormal Operators

These operators were defined in the Introduction; they are operators having 0 as a best normal approximant. This class of operators is not vacuous: indeed, an argument given in [3, p. 271] shows that any singular norm-one operator possessing a left inverse of norm $\leqslant 1$ is antinormal. Thus any nonunitary isometry is antinormal. The unilateral shift being an immediate example, we see that an antinormal operator can be quasinormal (and hence subnormal, hyponormal, etc.). Since the adjoint of an antinormal operator is again antinormal, we may conclude that every (nonunitary) maximal partial isometry (together with all their nonzero scalar multiples) is antinormal. Thus, recalling Kadison's characterization [8] of the extreme points of the unit ball of certain operator algebras (cf. in particular [3, p. 265]), we may assert that every extreme point of the unit ball of $\mathcal{B}(H)$ is either normal (actually unitary) or else antinormal.

We now establish a sufficient condition for an operator $T \in \mathcal{B}(H)$ to be antinormal. The condition requires that the distance between T and the unitary subgroup $\mathcal{U}(H)$ of $\mathcal{B}(H)$ be as large as possible. Before stating the theorem precisely, it is convenient to isolate a portion of its proof as a lemma. This lemma actually tells us a little more than we need to know for our immediate purpose: however, it may have some independent interest. It asserts that the unit ball of $\mathcal{N}(H)$ consists of all averages of commuting unitary operators.

Lemma. *Let $T \in \mathcal{B}(H)$ with $\| T \| \leqslant 1$. Then T is normal if and only if there exist commuting unitaries U and V such that $T = (U + V)/2$.*

Proof. If we have any pair A, B of commuting normal operators, then $AB^* = B^*A$, $A^*B = BA^*$ (as follows, for example, by the Fuglede Theorem [3, p. 98]), and consequently $A + B$ is normal. If conversely we have $T \in \mathcal{N}(H)$ with $\| T \| \leqslant 1$, then we have the polar decomposition $T = WP$, where $W \in \mathcal{U}(H)$, $P \in \mathcal{P}(H)$, $\| P \| \leqslant 1$, and $WP = PW$. Now, as is well known, we can write $P = (W_1 + W_1^*)/2$, where W_1 is the unitary $P + i(I - P^2)^{1/2}$. Since W_1 is a function of P, it commutes with W. Thus,

$$T = WP = (WW_1 + WW_1^*)/2,$$

expresses T in the desired manner.

THEOREM 2. *Let $T \in \mathscr{B}(H)$ satisfy*

$$\operatorname{dist}(T, \mathscr{U}(H)) = 1 + \| T \|. \tag{2}$$

Then T is antinormal.

Proof. We note that the left-hand side of (2) is always $\leqslant$ the right-hand side for any $T \in \mathscr{B}(H)$. Now let N be any normal operator. By the convexity of the function $\alpha \mapsto \| T - \alpha N \|$ it will suffice to assume that $\| N \| \leqslant \| T \|$ and show that $\| T - N \| \geqslant \| T \|$. By the lemma there exist (commuting) unitaries U and V such that

$$N = \tfrac{1}{2} \| T \| (U + V).$$

Suppose that $\| T - N \| < \| T \|$. Then

$$\begin{aligned} \| T - \| T \| U \| &= \| T - \tfrac{1}{2} \| T \| U - \tfrac{1}{2} \| T \| U - \tfrac{1}{2} \| T \| V + \tfrac{1}{2} \| T \| V \| \\ &\leqslant \| T - \tfrac{1}{2} \| T \| (U + V) \| + \tfrac{1}{2} \| T \| \| U - V \| \\ &< 2 \| T \|. \end{aligned}$$

Now if $\| T \| \geqslant 1$, then

$$\begin{aligned} \| T - U \| &\leqslant \| \operatorname{sgn}(T) - U \| + \| T - \operatorname{sign}(T) \| \\ &< 2 + | \, \| T \| - 1 \, | = 1 + \| T \|, \end{aligned}$$

contradicting (2). On the other hand, if $\| T \| \leqslant 1$, then

$$\begin{aligned} \| T - U \| &= \| T - (\| T \| + (1 - \| T \|)) \, U \| \\ &< 2 \| T \| + | \, 1 - \| T \| \, | = 1 + \| T \|, \end{aligned}$$

which again contradicts (2). Q.E.D.

It was shown in [3, p. 275] that the unilateral shift satisfies the condition of Theorem 2, and the same proof applies to any other nonunitary isometry. In this way we see again that all maximal partial isometries are either normal (actually unitary) or else antinormal. We conjecture that in fact any (non-normal) *subnormal* partial isometry satisfies the condition (2) and is thereby antinormal. (It follows from Theorem 1 above and Theorem 5 of [6] that any such operator is either antinormal or else at distance $\frac{1}{2}$ from $\mathscr{N}(H)$.) However, in general a partial isometry will not be antinormal. In fact, the operator $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ was shown in Section 1 to be at distance $\frac{1}{2}$ from the two-dimensional normal operators (this fact was also pointed out in [6]).

Since we incline toward the belief that every (norm-one) antinormal operator must be a partial isometry, it is interesting to analyze this last example to see why it should fail to be antinormal. Two reasons appear; the operator is compact (see Theorem 3), and the final space of the operator is disjoint from its initial space. We now show that any partial isometry satisfying a slightly strengthened form of this condition not only fails to be antinormal, but actually possesses a better Hermitian approximant than 0.

EXAMPLE. Let T be a partial isometry whose final space F is disjoint from and makes a positive angle with its initial space M. We are going to show that $\| T - T^* \| < 2$. This will show that $\text{dist}(T, \mathscr{H}(H)) < 1 = \| T \|$; that is, there is actually a Hermitian operator within a distance less than unity from T. Let $\| x \| = 1$; then

$$\begin{aligned} \| T(x) - T^*(x) \| &= \| TP_M(x) - T^*P_F(x) \| \\ &\leqslant \| P_M(x) \| + \| P_F(x) \|, \end{aligned} \tag{3}$$

where P_M and P_F are the orthogonal projectors onto the indicated subspaces. Now choose $\delta > 0$ so that

$$\sqrt{2} > \alpha \equiv 2 \sin \tfrac{1}{2} \measuredangle (M, F) - (2\delta)^{1/2} > 0.$$

Suppose that $\| P_M(x) \| > 1 - \delta$. Then there is a unit vector $m \in M$ such that $\text{re}\langle x, m \rangle > 1 - \delta$. Hence $\| x - m \|^2 < 2\delta$. Now for any unit vector $f \in F$ we have

$$\| m - f \| \geqslant 2 \sin \tfrac{1}{2} \measuredangle (M, F)$$

(see, for example, [9, p. 28]). Therefore,

$$\begin{aligned} \| x - f \| &\geqslant \| m - f \| - \| x - m \| \\ &\geqslant 2 \sin \tfrac{1}{2} \measuredangle (M, F) - (2\delta)^{1/2} \equiv \alpha, \end{aligned}$$

and so from

$$\| x - \text{sgn}(P_F(x)) \|^2 = \| x - P_F(x) \|^2 + \| P_F(x) - \text{sgn}(P_F(x)) \|^2$$

and

$$1 = \| P_F(x) \|^2 + \| x - P_F(x) \|^2 = \| P_F(x) \|^2 + \text{dist}(x, F)^2,$$

we obtain

$$\| P_F(x) \| \leqslant 1 - \tfrac{1}{2}\alpha^2.$$

Thus we see from (3) that

$$\begin{aligned} \| T - T^* \| &= \sup_{\| x \| = 1} \| T(x) - T^*(x) \| \\ &\leqslant \max\{2 - \delta, 2 - \tfrac{1}{2} \alpha^2\} < 2, \end{aligned}$$

Q.E.D.

What is required of an operator in order that it be antinormal? It may be necessary that T be a partial isometry (as suggested previously) and/or satisfy the condition of Theorem 2. While unable at present to fully characterize antinormal operators, we show next certain operators can be excluded from consideration.

THEOREM 3. *Let* T *be an invertible operator in* $\mathscr{B}(H)$. *Then with* $|T| = (T^*T)^{1/2}$,

$$\operatorname{dist}(T, \mathscr{N}(H)) \leqslant \tfrac{1}{2} \operatorname{diam} \sigma(|T|), \tag{4}$$

so that T *is not antinormal. Consequently, no compact operator can be antinormal.*

Proof. If T is invertible, we have the polar decomposition $T = U|T|$, with U unitary. Let $\lambda = \sup\{\alpha; \alpha \in \sigma(|T|)\}$, $\mu = \inf\{\alpha; \alpha \in \sigma(|T|)\}$. Then

$$\operatorname{dist}(T, \mathscr{N}(H)) \leqslant \| T - \tfrac{1}{2}(\lambda + \mu) U \| = \| |T| - \tfrac{1}{2}(\lambda + \mu) I \| \leqslant \tfrac{1}{2}(\lambda - \mu).$$

Now assume that T is compact with $\|T\| = 1$, say. Then there is a sequence of finite rank operators T_n with $\|T_n\| = 1$ and $\|T_n - T\| \to 0$. It will suffice to show that $\operatorname{dist}(T_n, \mathscr{N}(H)) \leqslant \frac{1}{2}$, for all n. Now each T_n is the direct sum of an operator T_n' with a finite dimensional domain, H_n say, and a zero-operator. It is easy to see that $\operatorname{dist}(T_n, \mathscr{N}(H)) \leqslant \operatorname{dist}(T_n', \mathscr{N}(H_n))$. But in $\mathscr{B}(H_n)$, there is a sequence $S_{m,n}$ of norm-one invertible operators with limit T_n'. By (4), $\operatorname{dist}(S_{m,n}, \mathscr{N}(H_n)) \leqslant \frac{1}{2} \operatorname{diam}(|S_{m,n}|) \leqslant \frac{1}{2}$, whence $\operatorname{dist}(T_n', \mathscr{N}(H_n)) \leqslant \frac{1}{2}$ also. Q.E.D.

Note added in proof. Since this paper was submitted, a manuscript entitled "Proximinal sets of operators" by D. D. Rogers has appeared, wherein it is shown, among other things, that $\mathscr{N}(H)$ and $\mathscr{U}(H)$ are not proximinal unless, of course, H is finite dimensional.

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